From Energy-Time Uncertainty to Symplectic Excalibur

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Outline

1. Uncertainty Principle and Quantum Speed Limit
2. Displacement Problems in Symplectic Topology
3. Modeling with Derived Categories
4. Categorical Displacement Energy

Heisenberg Uncertainty Principle

- $\psi = \frac{\psi}{|\psi|} \in$ Hilbert space $\mathcal{V}$ pure quantum state
- $O \in \{$hermitian operators on $\mathcal{V}\}$ quantum observable
- $\langle O \rangle := \langle \psi | O | \psi \rangle$ expectation value
- $\triangle O := \sqrt{\langle O^2 \rangle - \langle O \rangle^2}$ standard deviation (uncertainty)

$\mathcal{V} = L^2(\mathbb{R}^n)$ suitable function space $\Rightarrow \triangle O_1 \triangle O_2 \geq \frac{1}{2}\langle [O_1, O_2] \rangle$
When $O_1 = \hat{q}_1$, $O_2 = \hat{p}_1 = \frac{\hbar}{i}\frac{\partial}{\partial q_1}$ (O_1 and O_2 are conjugated)

Heisenberg Uncertainty Principle:

$$\triangle \hat{q}_1 \triangle \hat{p}_1 \geq \frac{\hbar}{2} \quad \leftarrow \text{independent of } \psi$$

deduced from noncommutativity relation $[\hat{q}_1, \hat{p}_1] = i\hbar$.
Noncommutativity=**Obstruction for simultaneous measurement**.

Conservation Law

Noether Theorem: for conjugated physical quantities O_1, O_2, if the physical law does not depend on O_1, then the quantity O_2 is conserved under system evolution.

- Translation-Symmetry $\Rightarrow$ Conservation of Linear Momentum
- Rotation-Symmetry $\Rightarrow$ Conservation of Angular Momentum
- Phase-Symmetry $\Rightarrow$ Conservation of Charge
- Time-Symmetry $\Rightarrow$ Conservation of Total Energy

Heisenberg Uncertainty Principle is a statistical variant of Noether Conservation Law. So it is natural to ask

What is **Energy-Time Uncertainty** ?

Mysterious Energy-Time Uncertainty

When it comes to Energy-Time Uncertainty, the notion of simultaneous measurement becomes troublesome because:

1. Time is not a quantum observable
2. All observables can be measured with arbitrary accuracy in arbitrary short time

In the famous *Bohr-Einstein Debates*, Einstein demonstrated that fixed small $\triangle t$, we could measure E precisely using $E = mc^2$. But Bohr argued that the physical measurement of the mass m relies on a mechanical design against the gravity of Earth. Therefore, by General Relativity, such mechanical motion in the gravitational field yields an intrinsic uncertainty of time duration it experiences. (Still problematic, of course)

Quantum Speed Limit (Mandelstam-Tamm 1945)

The mysterious Energy-Time Uncertainty is not about simultaneous measurement nor Relativity, but rather **speed of quantum evolution!**

Let H quantum Hamiltonian operator and $P = |\psi\rangle\langle\psi|$ projector onto the state $\psi = \psi(t)$ then they satisfies

$$\triangle H \triangle P \geq \frac{1}{2}\langle [H, P] \rangle_{\psi(0)} \quad \& \quad \frac{dP}{dt} = \frac{i}{\hbar}[H, P]$$

Integrate against $t \in [0, \tau]$ and get

$$\tau \triangle H \geq \frac{\pi\hbar}{2} - \hbar \arcsin \sqrt{\langle P(\tau) \rangle}$$

Consider τ_{orth} such that $\psi(\tau) \perp \psi(0)$ then such τ_{orth} should satisfies

$$\tau_{orth} \geq \frac{\pi\hbar}{2} \frac{1}{\triangle H} := \tau_{QSL}$$

Quantum Speed Limit (Margolus-Levitin 1998)

$\triangle H$ may diverge. They showed that for $\langle H \rangle > 0$ with zero ground energy,

$$\tau_{orth} \geq \tau_{QSL} := \frac{\pi\hbar}{2}\max\{\frac{1}{\triangle H}, \frac{1}{\langle H \rangle}\}$$

1. Without referring to noncommutativity relation
2. τ_{QSL} sets an universal bound of **minimal time** for the system to evolves from one state to an orthogonal state with given **energy**
3. Being orthogonal = being distinguishable
4. τ_{QSL} sets an intrinsic scale for quantum computational capability
5. Recently (*Sci. Adv.*, 22 Dec 2021) both limits are tested for single atom in an optical trap

Beyond Quantum Mechanics (after 2018)

The formation

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Two surprising papers on Phys.Rev.Lett.:

1. QSL Is Not Quantum (Okuyama-Ohzeki)
2. QSL Across Quantum-Classical Transition (Shanahan-Chenu-Margolus-del Campo)

As many-particle effects they obtained **Classical Speed Limits** for Liouville equation and Wigner phase representation. Both QSL and CSL can be derived from **dynamical properties of Hilbert space under unitary system evolution**.

Mathematical Digest

Today:

An avatar of QSL/CSL in geometry and topology

Symplectic Manifold = Classical Phase Space

A **symplectic manifold** (M, ω) is a smooth manifold M with a closed non-degenerate 2-form ω. A symplectic map $(M_1, \omega_1) \xrightarrow{f} (M_2, \omega_2)$ is a smooth map with $f^*\omega_2 = \omega_1$.

- (Kähler type) $M =$ closed surface, $\omega =$ area form
- (Dynamical type) $M = T^*Q = \{(q, p) | q \in Q, p \in T_q^*Q\}$, $\omega = dq \wedge dp = \sum dq_i \wedge dp_i$

Darboux Theorem: for general symplectic (M, ω), near every point there is a coordinate chart **symplectic diffeomorphic** to open domain in $(T^*\mathbb{R}^n = \mathbb{R}^{2n}, dq \wedge dp)$. Therefore any two symplectic manifolds are locally equivalent.

- Symplectic geometry concerns about global problems.
- Problems for local charts can be very difficult.

Hamiltonian Dynamics = Physical Laws

Think of M a pool of physical states and ω a generator of physical laws. Given $F : (M, \omega) \times I \to \mathbb{R}$, we assign a (time-dependent) Hamiltonian vector field X_F by $dF(-) = \omega(X_F, -)$. The Hamiltonian dynamics $I \xrightarrow{x} M$ is the ODE

$$\dot{x}(s) = X_F(x(s))$$

In standard local coordinates $x = (q, p)$ it becomes

$$\dot{q}(s) = \frac{\partial F}{\partial p}(q, p, s) \quad \& \quad \dot{p}(s) = -\frac{\partial F}{\partial q}(q, p, s).$$

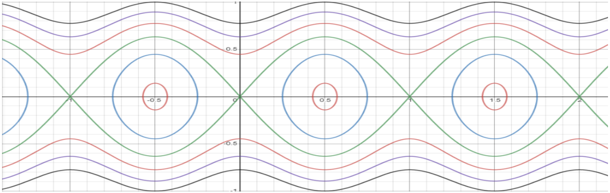
The solution of $\dot{x}(s) = X_F(x(s))$, when exists, is given by a flow

$$f_s : M \times I \to M.$$

Examples of Hamiltonian dynamics on T^*Q

Denote the function generating the flow by $F \rightsquigarrow f$.

1. $F(q,p,s) = C(s) \rightsquigarrow f_s = Id$ identity
2. $F(q,p) = \frac{1}{2}|p|^2 \rightsquigarrow f_s(q,p) = (q+sp,p)$ linear motion
3. $F(q,p) = \frac{1}{2}(|q|^2 + |p|^2)$ rotation
   $\rightsquigarrow f_s(q,p) = (\cos(s)q + \sin(s)p, -\sin(s)q + \cos(s)p)$.
4. $F(q,p) = 0.1\cos(2\pi q) + \frac{1}{2}|p|^2 \rightsquigarrow$ simple pendulum



Hamiltonian Diffeomorphisms

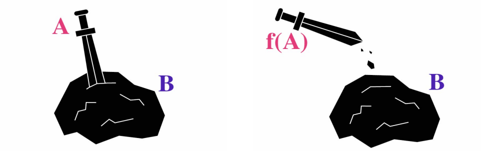
Each f_s is symplectic with $f_0 = Id$. If a symplectic isotopy can be obtained this way we call it Hamiltonian diffeomorphism. They are like exact differential 1-forms (since their dynamics are driven by dF) and are invisible by naive topological argument.

Time-independent Hamiltonian diffeomorphism form the most important class but they do not form a group. Let $F \rightsquigarrow f_s$ and $G \rightsquigarrow g_s$. Then their composition $f_s g_s \in Diff(M)$ is generated by $F(x,s) + G(f_s{}^{-1}x, s)$. So it is necessary to consider time-dependent Hamiltonian functions.

Proposition

All Hamiltonian diffeomorphisms form a subgroup $Ham(M,\omega)$ of $Diff(M)$.

Displacement Problem in Symplectic Topology



Given $A, B \subset (M, \omega)$. We say A is **displaceable** from B if $\exists f = (f_s)|_{s=1} \in Ham(M)$ such that

$$f(A) \cap B = \emptyset.$$

For example $M = S^2$, $A = B = S^1$

1. if A is a big circle then A is **not displaceable** from itself
2. if A is a small circle then A is **displaceable** from it self

Displacement Energy

More quantitatively, we define the **Hofer displacement energy** to be

$$e(A, B) := \inf_{F \rightsquigarrow f} \{\|F\| : f(A) \cap B = \emptyset\}$$

where the norm is defined by

$$\|F\| := \int_0^1 (\max_M F_s - \min_M F_s) ds.$$

which is L^1 in time and L^∞ in phase space.

When A is not displaceable from B we denote by $e(A, B) = \infty$.

Quantitative Displacement in Symplectic Topology

We say $A \subset (M, \omega)$ a **Lagrangian** submanifold if $\omega|_A = 0$. It follows that $\dim(A) = \frac{1}{2}\dim M$. Being Lagrangian is a sharp condition in the sense of symplectic displacement.

Some deep facts:

1. if $\dim A + \dim B < \dim M$ then $e(A, B) = 0$
2. if $\dim(A) = \frac{1}{2}\dim M$ but A is not Lagrangian and there is no topological obstruction, then $e(A, A) = 0$
3. if A, B are Lagrangian and $HF^\bullet(A, B) \neq 0$, then $e(A, B) = \infty$

On the other hand, let $B \subset M$ open then

$$e(B, B) > 0.$$

Quantitative Displacement of Mixed Type

When A is compact Lagrangian and B is open, our result is

Theorem

If $A \cap B \neq \emptyset$ *then* $e(A, B) > 0$.

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Theorem

If $A \cap B \neq \emptyset$ then $e(A, B) > 0$.

Moreover, we will use a **quantum-speed-limit** argument to characterize the energy $e(A, B)$ where we represent

- Lagrangian A by a **quantum-like state**
- Open B by a **collection of quantum-like states**
- Hamiltonian diffeomorphism f by a **unitary-like system evolution**
- Displacement by **orthogonality of states**

We ask for the **least energy** to displace a state from a given collection of states in a **unit of time**.

In absence of Hilbert space

While QSL/CSL is deduced from the unitary dynamic of Hilbert space, the function theory on symplectic manifold does not fit very well into L^2-space formalism.

In fact, **only L^∞ takes place**. It was proved by Eliashberg-Polterovich that if we replace the L^∞ norm in

$$\|F\| := \int_0^1 (\max_M F_s - \min_M F_s) ds.$$

by any L^p-norm, $1 \leq p < \infty$, then the corresponding distance on $Ham(M, \omega)$ is degenerate (and vanishing for closed M).

Modeling with Derived Categories

Fix a ground ring $\mathbb{k}$.
$D(Q)$ = derived category of sheaves of $\mathbb{k}$-modules over manifold Q.
(Sheaves contain a great deal of information!)
Work in a slightly refined dg-triangulated category $\mathcal{D}$ (described later).

1. $\mathcal{D}$ is our derived space of states
2. $Rhom : \mathcal{D}^{op} \times \mathcal{D} \to D(\mathbb{k}\text{-mod})$ is our derived inner product which measures mutual overlapping of states
3. Instead of orthogonality we have left/right semiorthogonality
4. $f \in Ham(T^*Q)$ induces autoequivalence $\mathcal{D} \to \mathcal{D}$.

Microsupport à la Kashiwara-Schapira

A bridge between algebra and geometry

$$SS : \{\text{sheaves}\} \to \{\text{sets}\}$$

For $\mathcal{F} \in D(Q)$, define its **microsupport** $SS(\mathcal{F}) \subset T^*Q$ by the closure of those (q_0, p_0) such that $\exists \phi : Q \xrightarrow{C^1} \mathbb{R}$ and $d\phi(q_0) = p_0$ satisfying

$$(R\Gamma_{\{q \in Q | \phi(q) \geq \phi(q_0)\}} \mathcal{F})_{q_0} \not\cong 0$$

$SS(\mathcal{F})$ is the closed conic subset consists of **singular codirections**, i.e., codirections along which the derived sections of $\mathcal{F}$ cannot propagate. This notion is genuinely **derived**: one cannot recover $SS(\mathcal{F})$ from $SS(H^\bullet(\mathcal{F}))$.

Subcategory as collection of states

For geometric reason instead of $D(Q)$ we choose to work in a more "faithful" category $\mathcal{D}(Q \times \mathbb{R})$. Let

$$\rho : T^*(Q \times \mathbb{R}) = \{(q, p, z, \zeta)\} \to \{(q, \frac{p}{\zeta})\} = T^*Q$$

and

$$D_{\zeta \leq 0} = \{\mathcal{F} \in D(Q \times \mathbb{R}) | SS(\mathcal{F}) \subset \{\zeta \leq 0\}\}$$

Definition (Tamarkin Category)

- $\mathcal{D} := D_{\zeta \leq 0}(Q \times \mathbb{R})^{left\perp}$ w.r.t. *Rhom* in $D(Q \times \mathbb{R})$
- $\mathcal{D}_A := \{\mathcal{F} \in \mathcal{D} | SS(\mathcal{F}) \subset \rho^{-1}(A)\}$, $\forall A \overset{\text{cls}}{\subset} T^*Q$
- $\mathcal{D}_B := {D_{T^*Q \backslash B}}^{\text{left}\perp}$ w.r.t *Rhom* in $\mathcal{D}$, $\forall B \overset{\text{open}}{\subset} T^*Q$

Interleaving Distance

Let $T_c : z \mapsto z + c$ acting on $D(Q \times \mathbb{R})$. A remarkable feature of $\mathcal{D}$ is we have for $a \geq 0$, a natural transformation between endofunctors

$$\tau_a : Id \Rightarrow T_a$$

Let $\mathcal{F}, \mathcal{G} \in \mathcal{D}(Q \times \mathbb{R})$ and $a, b \geq 0$. We say the pair $(\mathcal{F}, \mathcal{G})$ is (a, b)-**interleaving** if there exists morphisms $\mathcal{F} \underset{\delta}{\overset{\alpha}{\rightrightarrows}} T_a\mathcal{G}$ and $\mathcal{G} \underset{\gamma}{\overset{\beta}{\rightrightarrows}} T_b\mathcal{F}$ satisfying

$$\begin{cases} [\mathcal{F} \xrightarrow{\alpha} T_a\mathcal{G} \xrightarrow{T_a\beta} T_{a+b}\mathcal{F}] = \tau_{a+b}(\mathcal{F}) \\ [\mathcal{G} \xrightarrow{\gamma} T_b\mathcal{F} \xrightarrow{T_b\delta} T_{a+b}\mathcal{G}] = \tau_{a+b}(\mathcal{G}) \end{cases}$$

The **interleaving distance** is defined to be

$$d(\mathcal{F}, \mathcal{G}) := \inf\{a + b \,|\, (\mathcal{F}, \mathcal{G}) \text{ is } (a, b)\text{-interleaving}\}$$

Categorical Energy

Theorem (Existence of Microlocal Projector)

Let B be a bounded open subset of T^*Q, then in $\mathcal{D}(Q \times Q \times \mathbb{R})$ there exists an exact triangle $(\mathcal{P}_B \to \mathbb{K}_\Delta \to \mathcal{Q}_B \xrightarrow{+1})$ such that the convolution with the above triangle gives rise to the semiorthogonal decomposition with respect to the triple of subset categories $(\mathcal{D}_B(Q \times \mathbb{R}), \mathcal{D}(Q \times \mathbb{R}), \mathcal{D}_{T^*Q \setminus B}(Q \times \mathbb{R}))$. Moreover, the projector construction $B \mapsto \mathcal{P}_B$ is compatible with Hamiltonian dynamics.

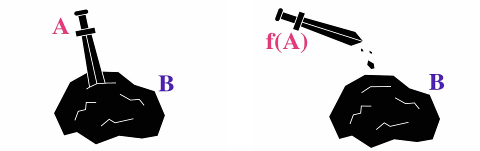
$\mathcal{P}_B$ admits a **right-adjoint** functor $\mathcal{E}_B$:

$$Rhom(\mathcal{G} \bullet \mathcal{P}_B, \mathcal{F}) \cong Rhom(\mathcal{G}, \mathcal{E}_B(\mathcal{F}))$$

Definition (Categorical Energy relative to B)

$\forall \mathcal{F} \in \mathcal{D}(Q \times \mathbb{R})$ we define $e_B(\mathcal{F}) := d(0, \mathcal{E}_B(\mathcal{F}))$.

Displacement Energy of Symplectic Excalibur



Recall the definition of Hofer displacement energy

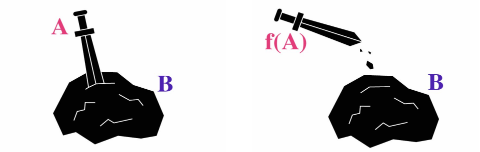
$$e(A, B) := \inf_{F \rightsquigarrow f} \{ \int_0^1 (\max_M F_s - \min_M F_s) ds \; : \; f(A) \cap B = \emptyset \}$$

Theorem (Comparison with Hofer Energy)

Given A closed and B open. Then for any $\mathcal{F} \in \mathcal{D}_A(Q \times \mathbb{R})$ one has

$$e(A, B) \geq e_B(\mathcal{F}).$$

Displacement Energy of Symplectic Excalibur



Theorem (Quantitative Displacement of Mixed Type)

Given A compact Lagrangian and B open. Assume that $A \cap B \neq \emptyset$, then

$$e(A, B) > 0$$

To prove this positivity it suffices to localize to a Darboux-Weinstein neighborhood T^*A of A and open ball B in $T^*\mathbb{R}^n \subset T^*A$.
Moreover, for suitable choice of $\mathcal{F} \in \mathcal{D}_A$ a nontrivial lower bound estimate (in the sense of hard analysis) of $e_B(\mathcal{F})$ is available.

Displacement Energy of Symplectic Excalibur

In T^*R^n, let $B = B(r) = \{q^2 + p^2 < r^2\}$ be a standand open ball and let $\mathcal{F} = \Bbbk_{\mathbb{R}^n \times \mathbb{R}_{\geq 0}} \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ be the sheaf quantization of the zero section $\mathbb{R}^n \times \{p = 0\}$. Our knowledge of $\mathcal{P}_B$ and $\mathcal{E}_B$ enables us to compute:

Proposition (Capacity-like Property)

$e_B(\mathcal{F}) \geq \frac{1}{2}\pi r^2$.

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Let L be a smooth manifold and suppose $U = \jmath(B(r))$ is a symplectically embedded ball of T^*L relative to L (that is $\jmath^{-1}(L) = \mathbb{R}^n \cap B(r)$).

Theorem (Relative Energy-Capacity Inequality)

$e(L, U) \geq \frac{1}{2}\pi r^2$.

Summary

Quantum Footprints in symplectic geometry and topology

1. **Uncertainty Principle** $\rightsquigarrow$ Nonsqueezing of Symplectic Balls
2. **Quantized Phase-Energy Levels** $\rightsquigarrow$ Nonsqueezing of Contact Balls
3. **Quantum Unsharpness** $\rightsquigarrow$ Rigidity of Partition of Unity
4. **Quantum Speed Limit** $\rightsquigarrow$ Symplectic Displacement Energy

Question

Pick your favorite quantum-like phenomenon.
Can you find its footprint in symplectic geometry and topology?

Thank You !